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Cascading quivers from decaying D-branes

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Abstract: We use an argument analogous to that of Kachru, Pearson and Verlinde to argue that cascades in $L^{a,b,c}$ quiver gauge theories always preserve the form of the quiver, and that all gauge groups drop at each step by the number M of fractional branes. In particular, we demonstrate that an NS5-brane that sweeps out the S^3 of the base of $L^{a,b,c}$ destroys M D3-branes.

Keywords: AdS-CFT Correspondence, Duality in Gauge Field Theories, Brane Dynamics in Gauge Theories.

Contents

1. Introduction

Klebanov and his collaborators have demonstrated [1 − 4] that type IIB string theory on $AdS_5 \times T^{1,1}$ is holographically dual to a cascading gauge theory. Three years later it was conjectured that there are more cascading gauge theories dual to the product of AdS_5 and an infinite family of Sasaki-Einstein manifolds $L^{a,b,c}$ that generalise $T^{1,1}$ [5 − 18]. However these cases are complicated by the fact that multiple gauge couplings become strong simultaneously, and so one does not quite know how to perform the duality, for example it may be that for a fixed $L^{a,b,c}$ there exist a network of walls separating domains of initial values of gauge couplings which exhibit different cascades [19 − 21]. We will argue that only one such pattern of cascades appears to be consistent with RR charge conservation in the dual gravity description in the case of compactifications on $AdS_5 \times L^{a,b,c}$ with a, b, c and $d \equiv a + b - c$ are relatively prime and also in the cases with $c = d$, in which $L^{a,b,c}$ is a $Y^{p,q}$ [12].

The manifolds $L^{a,b,c}$ are similar to $T^{1,1}$, the base of the conifold. Topologically they are identical, if a, b, c and d are relatively prime then $L^{a,b,c}$ is diffeomorphic to $T^{1,1}$ and both are diffeomorphic to $S^2 \times S^3$ [22, 1]. An explicit diffeomorphism relating $T^{1,1}$ to $S^2 \times S^3$ was presented in [23]. However the metrics on $T^{1,1}$ and $L^{a,b,c}$ are not equivalent and as a result the world-volume gauge symmetries of the gauge theories dual to the $AdS_5 \times T^{1,1}$ and $AdS_5 \times L^{a,b,c}$ backgrounds are very different. The gauge theories dual to $L^{a,b,c}$ compactifications are far more complicated and their vacuum structures are not understood, which is an obstruction to the analysis of their cascades of Seiberg dualities.

While it remains quite difficult to determine the vacuum structure of these theories, we will argue that the topology (in fact just the homology) of $L^{a,b,c}$ along with the fluxes present in the compactification already places a strong constraint on the dualities allowed in the dual gauge theory. This constraint arises by imposing that the dualities arise from processes that conserve RR charge, generalising the NS5-brane nucleation in the $T^{1,1}$ case

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which was presented in [24 – 26]. More specifically, consider $T^{1,1}$, which we recall again is diffeomorphic to $S^2 \times S^3$, with M units of RR 3-form flux F_3

$$\int_{S^3} F_3 = M. \tag{1.1}$$

The dual gauge theory, which intuitively lives on N D3-branes that are put at points on $L^{a,b,c}$, has a $\mathrm{SU}(N) \times \mathrm{SU}(N+M)$ gauge symmetry. Now consider an NS5-brane that wraps the 4-dimensional horizon and also wraps a contractible 2-sphere at fixed latitude θ in the S^3 of $T^{1,1}$. There is a family of such configurations, parameterised by the latitude θ. The central result of [24, 25] is that the parameter θ parameterises the re-normalisation group direction, and in particular as the gauge theory flows into the IR the 2-sphere nucleates at the south pole of the S^3, moves up to the equator and then shrinks down to nothing again at the north pole. We will refer to this process as a MMS instanton, as it was first described in [27]. A somewhat simplified version of this system was analysed classically in [28].

When the NS5-brane shrinks down to nothing there are M (anti) D3-branes left. One can see this immediately using RR charge conservation. The NS5-brane sources H flux, and by Gauss' law the MMS instanton increases the H flux by one Dirac unit. The total RR 3-charge is equal to the sum of the brane contribution, equal to the number of D3-branes, plus a bulk contribution

$$Q_{RR} = N_{D3} + \int H \wedge F_3. \tag{1.2}$$

When H increases by a single unit, this wedge product increases by M units, as one finds for example by using Poincaré duality to express the integral of $\Delta H \wedge F_3$ as the integral (1.1) of F_3 over the S^3 swept out by the NS5. The total RR 3-charge must be conserved, and so if the bulk charge increases by M units then the brane charge must decrease by M units, meaning that there are M less D3-branes, leaving $N - M$. The new gauge group is then $\mathrm{SU}(N - M) \times \mathrm{SU}(N - M + M) = \mathrm{SU}(N - M) \times \mathrm{SU}(N)$. Alternately one may use the NS5-brane worldvolume theory to see that the instanton leaves M anti-branes. The worldvolume Wess-Zumino term

$$S_{\mathrm{NS5}} \supset \int C_2 \wedge C_4 \tag{1.3}$$

implies that C_2 is an electric source for the RR 4-form connection C_4, in other words, it carries D3-brane charge. Using (1.1) and Stoke's theorem one finds that the integral of C_2 over the 2-sphere wrapped by the NS5-brane decreases by M units during the MMS instanton, and so the D3-brane charge decreases by M units, leaving M anti-branes when the NS5 finally collapses.

In the $T^{1,1}$ case, only one simple gauge group is strongly coupled in the IR, which allows one to treat the other as a global symmetry and so find the Seiberg duality directly in the field theory, choosing the root of the baryonic branch in the sense of [29] and thus demonstrating that the duality is not only allowed by RR charge conservation but actually provides a weakly coupled description of the strongly coupled IR gauge theory. In general it is not certain that an allowed transition provides another description, and even if it does then there is no reason to believe that such a description should always be weakly coupled. We will not be so ambitious.

The MMS instanton is easily generalised to $L^{a,b,c}$, without recourse to the details of the $L^{a,b,c}$ gauge theory. One need only know that $L^{a,b,c}$ is diffeomorphic to $S^2 \times S^3$ and that (1.1) still holds, which is a consequence of the fact that the D5-branes wrap the S^2 which has intersection number one with the S^3. The above argument then implies that D3-brane charge is preserved modulo M, and so cascades are allowed which change the number N of D3-branes by an integral multiple of M. In particular this only leaves room for a single family of cascades, and so appears to exclude duality walls for example. We will see that the allowed cascades all preserve the form of the corresponding quiver.[1] One may object that these cascades only describe baryonic vacua, and that RR charge must also be preserved in the dual descriptions of the mesonic vacua which correspond to distinct dual theories, however these vacua are dual to topologically distinct compactifications and so escape.

In section 2 we review Seiberg dualities in cascading quivers by considering the specific example of the $Y^{2,1}$ gauge theory dual. Then in section 3 we will demonstrate that there exists a 3-form on $L^{a,b,c}$ that satisfies (1.1) and is $(2,1)$, as is required by supersymmetry [31, 32]. We then describe the generalization of the construction of the NS5-brane on the deformed conifold in ref. [24] to the case of a general L^{abc} compactification. We conclude in section 4 with comments on the generalisation to compactifications on other spaces and D7-brane processes, and possible resolutions to the apparent contradiction with the duality wall literature.

2. Seiberg duality in quiver gauge theories

The superconformal quiver gauge theories dual to $Y^{p,q}$ spaces were first constructed in [7] generalising the specific case of $Y^{2,1}$ [33, 34], and this was later further generalised to include the dual quivers for all $L^{a,b,c}$ spaces [10, 11, 13]. Cascading Seiberg dualities in these field theories were discussed by many authors, and the supergravity duals of these cascades have also been considered [35, 36, 14, 37].

The discussion of cascades in these gauge theories is more involved than in the familiar case of the conifold, because in the latter the quiver diagram has only two nodes. One of these two nodes is strongly coupled in the IR (the one-loop beta function is positive), while the other one is weakly coupled. So the choice of the node on which one should Seiberg dualize is clear in the gauge theory picture. Indeed, after the duality, we end up with the same node structure that we started with, but with shifted ranks for the gauge groups, and the process continues all the way to the base of the cascade, where we lose one of the nodes (at least when N is a multiple of M) and the cascading comes to an end, resulting in chiral symmetry breaking [4]. This ties in well with the picture presented by the Klebanov-Strassler supergravity solution dual to the gauge theory: there, one finds a

[1] This ties in well with the expectations from the dimer models [30, 11].

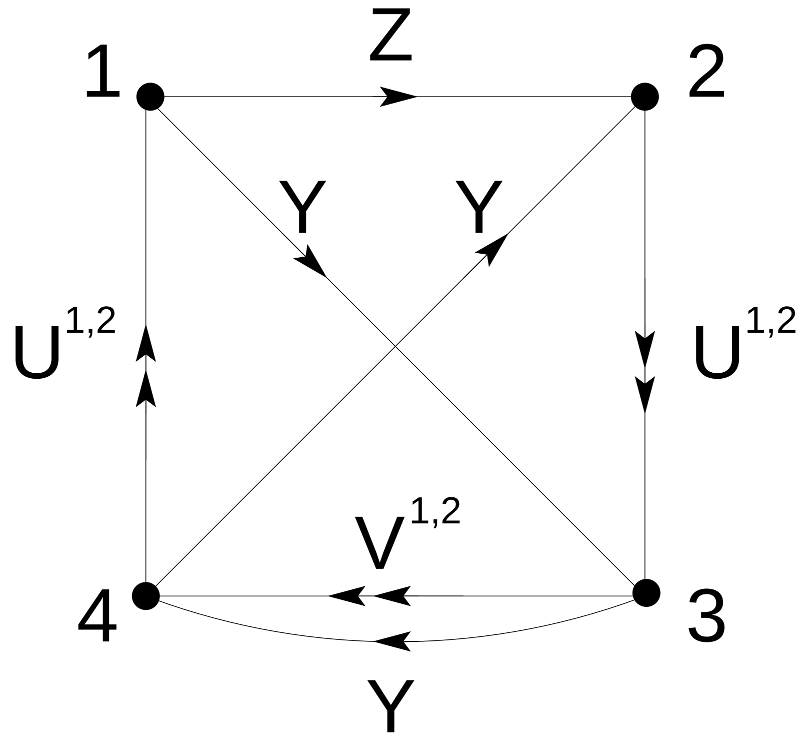

Figure 1: The quiver diagram for $Y^{2,1}$ theory.

radial dependence of the 5-form flux, which results in a logarithmic running of the effective number of D3-branes.

But in the case of the $L^{a,b,c}$ quivers, the situation is much less clear, and the choice of the cascade step could depend, in principle, on which node one chooses to dualize on. We will explain this in more detail by using the specific example of the $Y^{2,1}$ quiver in the reminder of this section. One purpose of this paper is to present a dual geometrical argument that gives us a natural way to choose the "right" cascade, dual to the supergravity description.

Let us now turn to the explicit example of $Y^{2,1}$. The quiver diagram for the theory is shown in figure 1. The diagram represents an $\mathcal{N} = 1$ gauge theory, where each node corresponds to a gauge group, and each arrow is a chiral bi-fundamental superfield, denoted by U^α, V^α, Y and Z in the figure, $\alpha = 1, 2$. If the gravity description corresponds to the simplest case, namely that of N D3-branes probing the apex of the cone over $Y^{2,1}$, then the worldvolume theory on the D3-branes is superconformal, and all the gauge groups are $\mathrm{SU}(N)$. To trigger the RG-flow that results in the cascade, we add M D5-branes and break the conformal invariance. It turns out that this changes the gauge groups to

$$\mathrm{SU}(N)_1 \times \mathrm{SU}(N+M)_3 \times \mathrm{SU}(N+2M)_4 \times \mathrm{SU}(N+3M)_2, \tag{2.1}$$

where the subscripts serve as a book-keeping device to keep track of the node associated to the corresponding gauge group in our quiver diagram. The superpotential for the theory is the sum of all the gauge invariant cubic and quartic operators in the fields listed above. For the case at hand,

$$W \sim \epsilon_{\alpha\beta} U_{41}^{\alpha} Y_{13} V_{34}^{\beta} + \epsilon_{\alpha\beta} V_{34}^{\alpha} Y_{42} U_{23}^{\beta} + \epsilon_{\alpha\beta} U_{23}^{\alpha} Y_{34} U_{41}^{\beta} Z_{12}. \tag{2.2}$$

The one-loop NSVZ beta function for the running gauge couplings for the various nodes can be computed from the usual formula,

$$\beta_i \equiv \frac{d(8\pi^2/g_i^2)}{d\log\mu} = \frac{3T(G) - \sum_i T(r_i)(1-2\gamma_i)}{1 - \frac{g_i^2}{8\pi^2} T(G)}. \tag{2.3}$$

Following Klebanov-Strassler and ignoring the denominator, and using the relation $\gamma_i = \frac{3}{2}R_i - 1$ relating anomalous dimensions and R-charges, we find the following beta functions for the various nodes:

$$\begin{aligned}
\beta_1 &= 3M + \frac{3M}{2}\big[6(R_U-1) + 2(R_Y-1) + 4(R_Z-1)\big] \\
\beta_2 &= 12M + \frac{3M}{2}\big[4(R_U-1) + 3(R_Y-1) + (R_Z-1)\big] \\
\beta_3 &= 6M + \frac{3M}{2}\big[6(R_V-1) + 8(R_U-1) + 4(R_Y-1)\big] \\
\beta_4 &= 9M + \frac{3M}{2}\big[4(R_V-1) + 2(R_U-1) + 6(R_Y-1)\big].
\end{aligned}$$

For each node, the gauge groups on the other nodes act as effective flavours. In the calculation, we have used the fact that $R_U + R_V + R_Y = 2$ and $2R_U + R_Y + R_Z = 2$, which are determined from the conditions for conformality when there are no D5-branes.

We can look up the R-charges of the various fields in [6], and the result is (for the specific case of $Y^{2,1}$):

$$R_Y = \frac{(-9+3\sqrt{13})}{3},\ R_Z = \frac{(-17+5\sqrt{13})}{3},\ R_U = \frac{4(4-\sqrt{13})}{3},\ R_V = \frac{(-1+\sqrt{13})}{3}. \tag{2.4}$$

From these explicit values, it follows immediately that nodes 2 and 4 are both strongly coupled in the infrared, unlike the case of the conifold where there was only one gauge coupling that blew up as we flowed down along the RG flow. So here the choice of the node to Seiberg dualize is in general initial condition dependent. But as observed in [35], if we choose to dualize on the node with the largest number of colours (in our case, this would be node 2), we end up with a quiver that is self-similar to the original one, and the usual logic of the cascade still goes through. This choice has a natural interpretation in terms of the D-brane decays that are allowed by K-theory in the dual geometry, it relates D-branes wrapping distinct homology classes that represent the same twisted K-theory class, as in [25, 38].

It is also important to note that with this choice of the node, the form of the superpotential is also unchanged after Seiberg duality, as we will now quickly demonstrate. To Seiberg dualize around node 2, we introduce meson fields $M_{43}^{\alpha} \equiv Y_{42}U_{23}^{\alpha}$ and $N_{13}^{\alpha} \equiv Z_{12}U_{23}^{\alpha}$ corresponding to the branches 1-2-3 and 4-2-3 that pass through node 2, and dual quarks $\tilde{q}_{24}, \tilde{q}_{21}, q_{32}^{\beta}$ corresponding to the legs that start at node 2. The superpotential for the dual theory with these fields will have the pieces (2.2) written in terms of the new fields, plus the pieces that couple the mesons and the dual quarks as dictated by the recipe for Seiberg duality:

$$W_{\text{temp}} \sim \epsilon_{\alpha\beta} U_{41}^{\alpha} Y_{13} V_{34}^{\beta} + \epsilon_{\alpha\beta} V_{34}^{\alpha} M_{43}^{\beta} + \epsilon_{\alpha\beta} U_{41}^{\alpha} N_{13}^{\beta} Y_{34} + \epsilon_{\alpha\beta} \tilde{q}_{24} M_{43}^{\alpha} q_{32}^{\beta} + \epsilon_{\alpha\beta} \tilde{q}_{21} N_{13}^{\alpha} q_{32}^{\beta}. \quad (2.5)$$

The VM-term is a mass term, and since we are after the IR physics, we integrate it out by setting

$$\frac{\partial W}{\partial V_{34}^{\alpha}} = 0 \implies U_{41}^{\alpha} Y_{13} = M_{43}^{\alpha},$$
$$\frac{\partial W}{\partial M_{43}^{\beta}} = 0 \implies V_{34}^{\alpha} = q_{32}^{\alpha} \tilde{q}_{24}.$$

The superpotential now looks like

$$W_{\text{new}} \sim \epsilon_{\alpha\beta} U_{41}^{\alpha} N_{13}^{\beta} Y_{34} + \epsilon_{\alpha\beta} \tilde{q}_{21} N_{13}^{\alpha} q_{32}^{\beta} + \epsilon_{\alpha\beta} U_{41}^{\alpha} Y_{13} q_{32}^{\beta} \tilde{q}_{24}, \quad (2.6)$$

which (after some identifications) is of the same form as (2.2).

The crucial thing to notice is that this works only if we choose to dualize on node 2. If we choose to dualize on node 4 (which we have seen is also strongly coupled), the resulting quiver (and the superpotential) is not of the same form as the one that we started with. It is straightforward to see this by Seiberg dualizing on node 4, the gauge groups become

$$\mathrm{SU}(N)_1 \times \mathrm{SU}(N+M)_3 \times \mathrm{SU}(2N+M)_4 \times \mathrm{SU}(N+3M)_2, \quad (2.7)$$

which is inconsistent with the original structure of the quiver. So the choice of the node is crucial for the cascade to work.

3. The cascade step from the $L^{a,b,c}$ geometry

The goal of this section is to demonstrate that, as in the conifold case, for an arbitrary $L^{a,b,c}$ with co-prime[2] a, b, c and $d \equiv a+b-c$ there is a 3-cycle Σ satisfying (1.1) for $M=1$ (and therefore for any M):

$$\int_{\Sigma} F_3 = 1. \quad (3.1)$$

We will then use this result in the construction of the NS5-brane instanton.

We will pursue the following strategy. A Calabi-Yau cone over $L^{a,b,c}$ is actually a Kähler quotient $\mathbb{C}^4 /\!\!/ \mathrm{U}(1)$, namely a gauged linear σ-model (GLSM) with $\mathrm{U}(1)$ charges $(a, b, -c, -d)$ [33]. Let us denote the $\mathbb{C}^4$ coordinates by z_i with $i = 1, \ldots, 4$. For each i there is a 3-submanifold Σ_i in $L^{a,b,c}$ defined by $z_i = 0$. These 3-cycles are calibrated and therefore supersymmetric, while their volumes correspond to the R-charges of various fields

[2]For non co-prime (a, b, c, d) the corresponding $L^{a,b,c}$ space is singular and degrees of freedom at the singularity may become important, and so it no longer suffices to consider the topology alone. In that case, perhaps the equivariant homology might determine the cascade structure.

in the dual gauge theory.[3] We will explicitly show that for an arbitrary $L^{a,b,c}$ there are two 3-cycles Σ_3 and Σ_4 satisfying:

$$\int_{\Sigma_3} F_3 = c \qquad \text{and} \qquad \int_{\Sigma_4} F_3 = d. \tag{3.2}$$

Since c and d are co-prime the Euclidean equation $nc + md = 1$ always has a solution. Finally, using the integers m and n we can construct a linear combination of Σ_3 and Σ_4 satisfying (3.1). As we have already mentioned, for $c = d$ the $L^{a,b,c}$ geometry reduces to $Y^{p,q}$ and we refer the reader to [35] for the detailed calculation in this case. We will briefly address this case at the end of the section.

The $L^{a,b,c}$ geometry can be briefly summarised as follows. The Sasaki-Einstein metric is given by:

$$ds_5^2 = ds_4^2 + (d\psi' + A)^2, \tag{3.3}$$

where the 4-dimensional metric is:[4]

$$ds_4^2 = \frac{(\eta-\xi)}{2F(\xi)}d\xi^2 + \frac{2F(\xi)}{(\eta-\xi)}(d\Phi + \eta d\Psi)^2 + \frac{(\eta-\xi)}{2G(\eta)}d\eta^2 + \frac{2G(\eta)}{(\eta-\xi)}(d\Phi + \xi d\Psi)^2, \tag{3.4}$$

with

$$F(\xi) = 2\xi(\alpha-\xi)(\alpha-\beta-\xi) \qquad \text{and} \qquad G(\eta) = -2\eta(\alpha-\eta)(\alpha-\beta-\eta) - 2 \tag{3.5}$$

for constant α and β, and the 1-form A is:

$$A = -\frac{1}{2}\left((\eta+\xi)d\Phi + \eta\xi d\Psi\right). \tag{3.6}$$

The coordinates η and ξ vary between two adjacent roots of the polynomials $F(\xi)$ and $G(\eta)$ respectively. In particular, $0 \le \xi \le \alpha - \beta$. The angular coordinates Φ and Ψ are defined by:

$$\Phi \equiv \frac{\psi}{2\beta} \qquad \text{and} \qquad \Psi \equiv \frac{1}{\alpha-\beta}\left(\frac{\phi}{2\alpha} - \frac{\psi}{2\beta}\right), \tag{3.7}$$

where both ϕ and ψ are 2π-periodic. The regularity of the entire 5-dimensional metric imposes a complicated relation between the constants α and β.

Now let us address the RR 3-form F_3. For the $10d$ solution to be supersymmetric, the form $\Omega \equiv H_3 - iF_3$ has to be $(2,1)$ [31, 32]. Moreover, on a Calabi-Yau cone over a Sasaki-Einstein space, the $(2,1)$-form is necessarily of the form:

$$\Omega_{(2,1)} = K\left(\frac{dr}{r} + i(d\psi' + A)\right) \wedge \omega_{(1,1)}, \tag{3.8}$$

where K is a constant and $\omega_{(1,1)}$ is a $(1,1)$ Kähler form on the 4-dimensional base of the 5-dimensional SE metric (3.3). For $L^{a,b,c}$ it is:

$$\omega_{(1,1)} = \frac{1}{(\eta-\xi)^2}\left(d(\eta-\xi)\wedge d\Phi + (\eta d\xi - \xi d\eta)\wedge d\Psi\right). \tag{3.9}$$

[3] See [39] for the relation between the 3-cycles and mesonic operators in the gauge theory.
[4] Here we adopt the notation of [14].

We will be interested in the 3-cycles Σ_3 and Σ_4, which correspond to $(\xi = 0, \phi = \text{const})$ and $(\xi = \alpha - \beta, \psi = \text{const})$ respectively. The integration over these cycles yields:

$$\int_{\Sigma_3} F_3 = K\frac{\pi}{\beta}\left(\frac{1}{\eta_2} - \frac{1}{\eta_1}\right)\Delta\psi' \quad \text{and} \quad \int_{\Sigma_4} F_3 = K\frac{\pi}{\alpha}\left(\frac{1}{\eta_2 - (\alpha-\beta)} - \frac{1}{\eta_1 - (\alpha-\beta)}\right)\Delta\psi', \tag{3.10}$$

where $\Delta\psi'$ is the period of ψ' and $\eta_{1,2}$ are the two adjacent roots of $G(\eta)$. Remarkably, these roots are related to the parameters α and β by:[5]

$$\frac{\alpha(\eta_2 - (\alpha-\beta))(\eta_1 - (\alpha-\beta))}{\beta\eta_2\eta_1} = \frac{c}{d}. \tag{3.11}$$

Thus setting

$$K = \frac{\beta}{\pi\Delta\psi'}\frac{\eta_1\eta_2}{\eta_2 - \eta_1}c \tag{3.12}$$

we arrive at (3.2), which in turn leads to (3.1) as we have already explained above. Remarkably, we could have considered the cycles Σ_1 and Σ_2 located at $\eta = \eta_1$ and $\eta = \eta_2$ respectively.

Finally, let us briefly review the $c = d$ case. In other words we have a $Y^{p,q}$ space with $p \equiv c$ and $q \equiv c - a = b - c$. Since the U(1) factor in the isometry group is now enlarged to SU(2) there are only three independent 3-cycles Σ_1, Σ_2 and Σ_3. These supersymmetric 3-cycles where investigated in [35]. It was found that for a certain value of the normalisation constant one obtains:

$$\int_{\Sigma_1} F_3 = p - q \qquad \int_{\Sigma_2} F_3 = p + q \qquad \text{and} \qquad \int_{\Sigma_3} F_3 = p, \tag{3.13}$$

which just reproduces our results for $c = d$. Furthermore, since p and $p - q$ (alternatively p and $p + q$) are co-prime we can use Σ_1 and Σ_3 to construct the 3-cycle Σ satisfying (3.1). This completes the proof of the main claim of the paper.

Now that we have established the existence of a 3-cycle supporting M units of flux, we may construct the NS5-brane instanton along the lines of the constructions in ref. [24]. In that paper, the authors construct the NS5-brane from two distinct points of view. First, they construct it from the point of view of the worldvolume theory on a stack of p anti-D3-branes, in the approximation that $p \leq\leq M$, which allows them to consider a flat space limit of the geometry, which exists as the geometry is nonsingular. In our case we may consider the nucleation of p anti-D3's and p D3's with any p, and so in particular we may consider a p which is much smaller than M and then use the same approximation. In this approximation they use S-duality to argue for the existence of a Myers dielectric term proportional to the dual B-field, which inflates the anti D3-branes into a spherical NS5-brane. They calculate the radius of the NS5-brane which minimizes the energy of the system, and then describe the MMS instanton as a tunneling process to another locally minimum energy solution. This argument is identical in our case.

Second, they consider the worldvolume theory of the spherical NS5-brane. Here they approximate the spacetime to be the product of S^3 and Minkowski space. The S^3 is

[5] In deriving this formula, we used (3.33) of [11] with $x_i = \alpha - \eta_i$ and the explicit form of $G(\eta)$ in (3.5).

homologically nontrivial in the deformed conifold. However in general no such deformation exists for an L^{abc}, and so the argument is more subtle in this case. Perhaps one may use a contractible S^3 which is far away from the singularity but links it.

4. Conclusions

$L^{a,b,c}$, at least when a, b, c and $d = a + b - c$ are relatively prime, is non-singular and diffeomorphic to $S^2 \times S^3$. In this note we have argued that this fact, together with RR charge conservation, is sufficient to restrict the form of possible cascades. We considered cascades in which each duality corresponds to an NS5-brane that sweeps out the 3-sphere, and argued that the 3-form RR flux on the 3-sphere implies that such a process necessarily destroys a number of D3-branes equal to the number of D5-branes, corresponding to a Seiberg duality in the gauge theory. We also checked that for any number of D5-branes there exists a 3-form representing the corresponding de Rham cohomology class which is $(2, 1)$, as is required by supersymmetry.

This result applies more generally. Only the integrals of the various forms over the cycles were important, and so it suffices to consider an integral sublattice of the de Rham cohomology, which in this case is isomorphic to the integral cohomology.[6] In particular, cascades caused by 5-branes sweeping out 3-cycles appear to never change the form of the quiver, because the 5-branes violate D3-brane charge which is classified by the zeroth cohomology of the compact space, which is always one-dimensional as the space is connected. Thus each step in the cascade corresponds to a change in a single parameter. If there are multiple 3-cycles, then the minimal cascade is simply the greatest common divisor of the number of D3-branes created by 5-branes wrapping the various 3-cycles. Exotic cascades may be possible if one also considers processes in which D7-branes nucleate, for example a D7-brane sweeping out a 5-cycle supporting a nontrivial H-flux will violate the D5-brane charge wrapping the 2-cycle dual to the H-flux in the 5-cycle. In practice many of these examples remain out of reach as they require an understanding of S-duality in the presence of D7-branes.

The self-similarity of these cascades appears to be in contradiction with the duality walls that are predicted from a purely gauge-theoretic point of view. It may be that this supergravity analysis is too naive, that one must consider also the physics at the tip of the cone, where many different cycles exists and may come in and out of existence via geometric transitions, however branes wrapping such cycles tend to lead to chiral anomalies in the gauge theory. Another possibility is that D7-brane processes must be considered in such cases. However, it may also be that in the gauge theory analysis, which relies on an analogy with a theory with a single simple gauge group, approximating the others to be global symmetries in the IR despite the sign of their beta functions, is invalid.

[6] In general the integral cohomology may also contain torsion subgroups, which may lead to interesting variations of the dual gauge theories corresponding to discrete torsion fluxes in the string theory compactification.

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